NOTES

Those Ubiquitous Archimedean Circles[1]

CLAYTON W. DODGE
University of Maine
Orono, ME 04469-5752

THOMAS SCHOCH
Krefelder Strasse 21
45145 Essen
Germany

PETER Y. WOO
Biola University
La Mirada, CA 90639

PAUL YIU
Florida Atlantic University
Boca Raton, FL 33431

The Bankoff triplet circle The *arbelos*, the figure formed by three mutually tangent semicircles with collinear centers and shown in FIGURE 1, has fascinated geometers since the time of the early Greeks. Also called the shoemaker's knife because it is shaped like that tool, it has been the subject of much study over the centuries. Many amazing and counterintuitive properties have been discovered in this figure, a few of which are described by Bankoff ([**1**] and [**2**]). That such a simple figure should be so rich is perhaps not so surprising since the arbelos is, after all, a triangle whose sides are semicircles.

Label the common diameter ACB and let the three semicircles be (O), (O_1), and (O_2) as shown in FIGURE 2. If one erects the common internal tangent line CD to the two interior circles, then the circles (W_1) and (W_2) inscribed in the resulting two regions ACD and BCD are called the *twin circles of Archimedes* and have the same radius. In 1974 Bankoff [**1**] pointed out that the twin circles of Archimedes are not

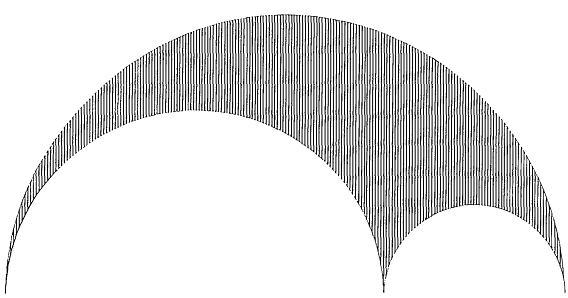

FIGURE 1
The arbelos.

[1]This paper is dedicated to the memory of Leon Bankoff.

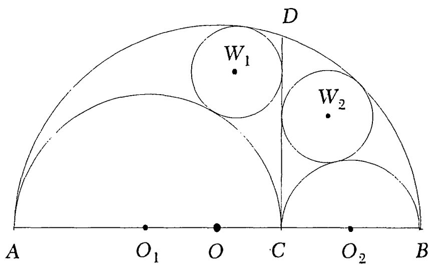

FIGURE 2
The twin circles.

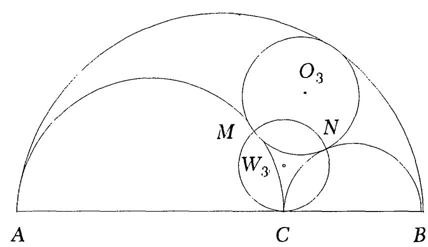

FIGURE 3
The Bankoff triplet circle.

twins, but two of triplets. That is, there is a third circle in the arbelos with the same radius. Inscribe circle (O_3) in the arbelos as shown in FIGURE 3. Then the circle (W_3) that passes through point C and the points of tangency M and N of circle (O_3) with circles (O_1) and (O_2) has the same radius as the twin circles. It is the *Bankoff triplet circle*. We denote certain circles congruent to the twin circles by (W_n) for positive integral n. Proofs of any of these assertions are postponed until after we note some other family members.

One might think that triplets are enough for any one household, but Bankoff discovered yet another member of that famous family. Let EF be the common external tangent to circles (O_1) and (O_2). *The Bankoff quadruplet circle* (W_4) is the circle inscribed in the circular segment of semicircle (O) and the chord EF (extended). See FIGURE 4. Furthermore, it is tangent to circle (O) at point D and is the smallest circle through point D and tangent to line EF. Now draw radius OD to cut EF at G. Then circle (W_4) has diameter GD.

The Dodge circles Bankoff and I (Clayton Dodge) discussed his discoveries, which led me to observe that if we drop a perpendicular CD' from point C to line OD, then $D'D$ is twice the diameter of an Archimedean circle. Furthermore, the two circles shown in FIGURE 5 include (W_4). We label the new circle, whose diameter is $D'G$, (W_5).

FIGURE 6 shows the translations of circles (W_1) and (W_2) that drop their centers onto the common diameter AB as circles (W_6) and (W_7), and the circle (W_8) on their centers as diameter. If these circles were merely translations, their interest would be quite low, but I found other reasons for their consideration. The common external

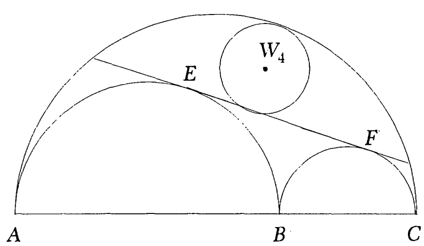

FIGURE 4
The Bankoff circle 4.

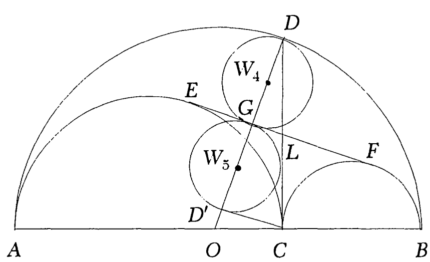

FIGURE 5
Circles 4 and 5.

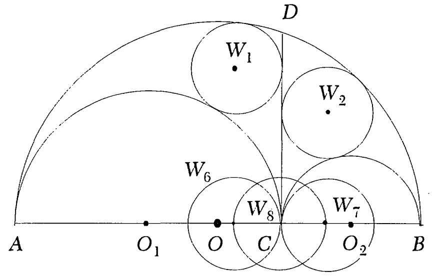

FIGURE 6
Circles 6, 7, and 8.

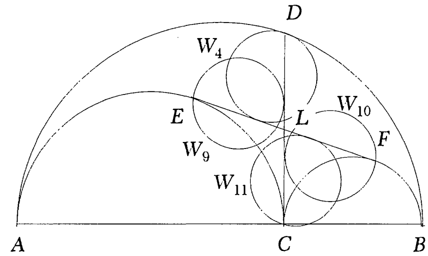

FIGURE 7
Circles 9, 10, and 11.

tangent to circles (O_1) and (W_7), for example, passes through point B and that for circles (O_1) and (W_8) passes through O_2. These properties will be examined more closely in FIGURE 25, near the end of this article.

Since segments CD and EF are equal and bisect each other at point L, there are three circles (W_9), (W_{10}), and (W_{11}) symmetric to (W_4), the smallest circle through D and tangent to EF. We take (W_9) to be the smallest circle through E and tangent to CD, (W_{10}) through F and tangent to CD, and (W_{11}) through C and tangent to EF. See FIGURE 7. Figures 5 and 7 show that circle (W_{11}) is also circle (W_5) translated through vector $\mathbf{D'C}$. Of course, circles (W_9) and (W_{10}) also translate down onto (W_6) and (W_7).

When I gave a lecture on these first eleven circles to a student group a few years ago, one of the students, Jonathan Dearing, pointed out circle (W_{12}), the circle whose diameter is the line of centers of circles (W_4) and (W_5), shown in FIGURE 8. Little did Archimedes realize the size of the family he uncovered! But we are not yet finished.

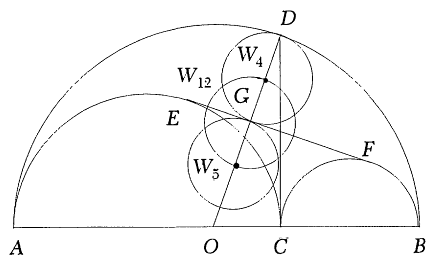

FIGURE 8
The Dearing circle 12.

Schoch's circles The development takes another turn at this point. In 1979 Martin Gardner wrote about Bankoff's triplet circle, inspiring the then student Thomas Schoch of Essen, Germany, to discover several more circles [**4**]. He sent his work, in German, to Gardner, who forwarded it to Bankoff, who was not familiar with German. Bankoff gave me a copy of it in 1996, when we were discussing the possibility of writing this article. Historically, then, Schoch's work precedes mine, but I shall continue the circle numbering as started above. I recognized the high quality of Schoch's paper and set out to locate him. He, still living in Essen, had not pursued his work on the circles until he found the arbelos website of Peter Woo [**5**] early in 1998. He then contacted Woo and told him of his findings. Paul Yiu led me to Woo, who had just completed a paper on his infinite family of Archimedean circles [**6**], and we all decided to combine our separate efforts into this paper.

FIGURE 9 shows Schoch's first two circles (W_{13}) and (W_{14}). They are found by drawing the circles $A(C)$, the circle with center A passing through point C, and $B(C)$ to cut circle (O) at points M_1 and M_2 respectively. Then (W_{13}) and (W_{14}) are the smallest circles through M_1 and M_2 and tangent to line CD. These circles, too, translate down to (W_6) and (W_7).

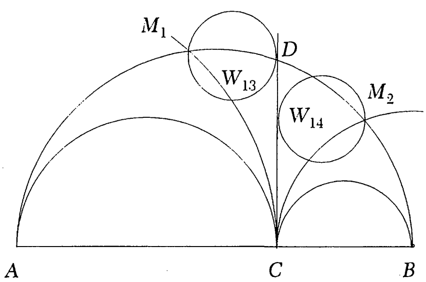

FIGURE 9
Schoch's circles 13 and 14.

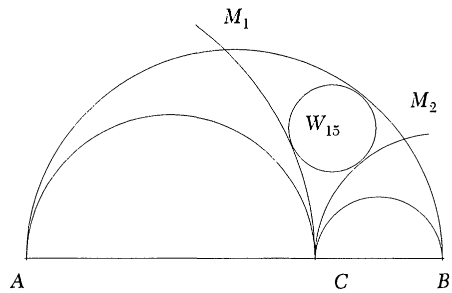

FIGURE 10
The Schoch circle 15.

Circle (W_{15}) is the incircle of curvilinear triangle CM_1M_2. See FIGURE 10. Next, drop a perpendicular $W_{15}K$ from point W_{15}, the center of circle (W_{15}), to line AB. Circle (W_{16}) is the circle centered on that perpendicular and tangent externally to circles (O_1) and (O_2), shown in FIGURE 11.

Let the line KW_{15} cut (O) at V. The smallest circle through V and tangent to the circle on O_1O_2 as diameter, which we denote by (O_1O_2), is circle (W_{17}). Let VK cut the circle (O_1O_2) at U. Then the circle through U, C, and K, denoted (UCK), that is, the circle (UC), is circle (W_{18}). See FIGURE 12.

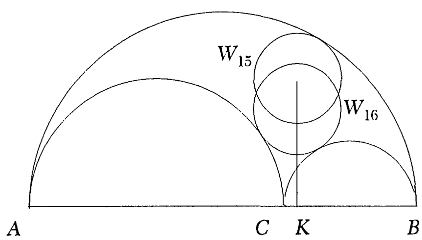

FIGURE 11
The Schoch circle 16.

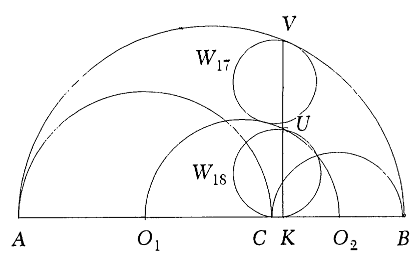

FIGURE 12
Schoch's circles 17 and 18.

If we construct on semicircle (O_1) an arbelos AC_1C similar to the given arbelos ACB, then the semicircle on C_1C as diameter is circle (W_6). Likewise we obtain circle (W_7) by constructing another similar arbelos CC_2B on CB as diameter. We let R_1, R_3, and R_4 be the highest points on circle (O_1), and on the two circles (AC_1) and (C_1C). Then $R_1R_3C_1R_4$ is a rectangle whose sides are in the ratio r_1/r_2. Similarly, $R_2R_5C_2R_6$ and RR_1CR_2 also are such rectangles, where R_2, R_5, R_6, and R are the highest points on circles (O_2), (CC_2), (C_2B), and (O). Furthermore, the lines C_1R_2, C_2R_1, and R_4R_5 all concur at a point Z on line VK. See FIGURE 13. In addition, since

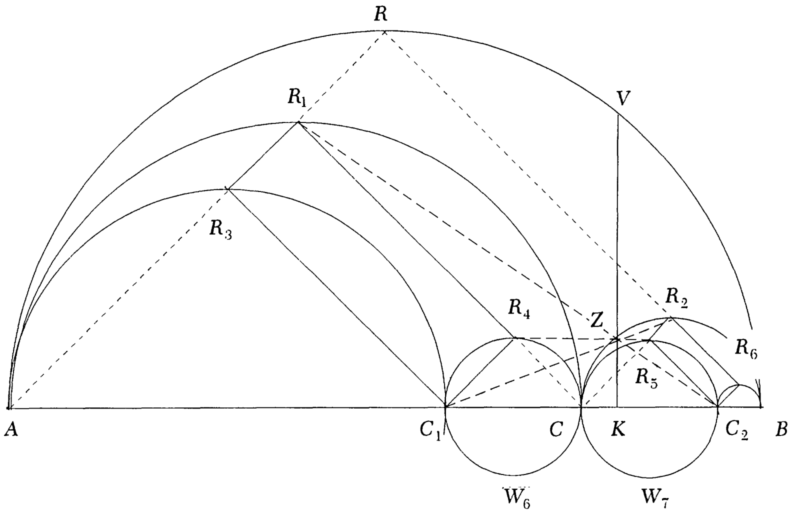

FIGURE 13
Circles 6 and 7 again.

the sides of these rectangles all make 45° angles with line AB, then points A, R_1, R_3, and R are collinear, as also are R_1, R_4, and C, and so forth. Paul Yiu **[8]** noted that the center W_3 of the Bankoff triplet circle lies at the intersection of the lines R_1O_2 and R_2O_1, thus providing an easy method for constructing that circle.

Locate point P on the circle on O_1O_2 as diameter so that a circle centered at P is externally tangent to both circles (O_1) and (O_2). Circle (W_{19}) is the smallest circle through point P and tangent to line AB. See FIGURE 14.

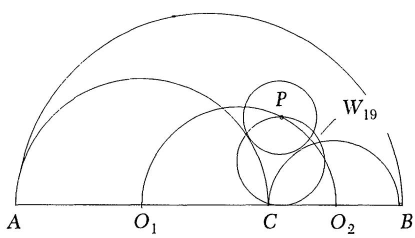

FIGURE 14
The Schoch circle 19.

Refer to FIGURE 15, where R_1, R_2, and R are the highest points on circles (O_1), (O_2), and (O) respectively. Then the circle (R') on R_1R_2 as diameter passes through O, C, and R. Let R_1R_2 cut CD at Y and OR at Q. Then the circles (RQ) and (YC) are circles (W_{20}) and (W_3). Circle (W_{21}) is the circle symmetric to (W_3) in line R_1R_2 and is tangent to circle (O) at point I, which is also the intersection of the circle (R_1R_2) and circle (O).

We note that points Z and K determine two more Archimedean circles, which we shall call (W_{22}) and (W_{23}), the circles $Z(K)$ and $K(Z)$, each centered on one of those points and passing through the other. Although Schoch did not mention these circles, he deserves the credit for them. FIGURE 16 shows these latest circles. Schoch also found the circles (W_4), (W_9), (W_{10}), and (W_{11}).

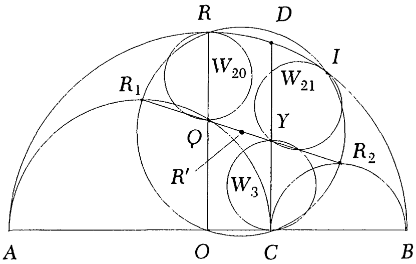

FIGURE 15
Schoch's circles 20 and 21.

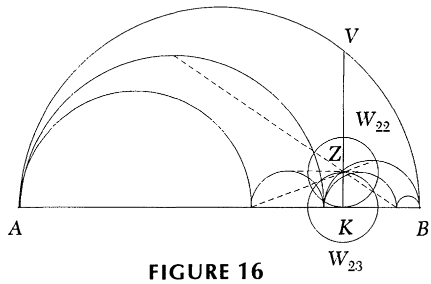

FIGURE 16
Circles 22 and 23.

Some loose ends In assembling proofs for all these circles, I observed three additional Archimedean circles, (W_{24}), (W_{25}), and (W_{26}). Circles (W_{24}) and (W_{25}) are centered at R_4 and R_5 respectively and pass through points W_6 and W_7 respectively. They are shown in FIGURE 17. FIGURE 18 shows circle (W_{26}), the circle symmetric to

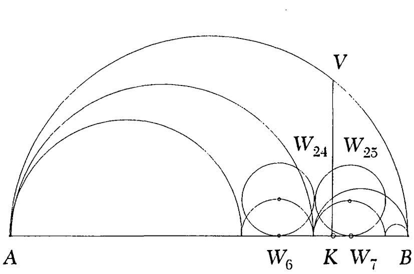

FIGURE 17
Circles 24 and 25.

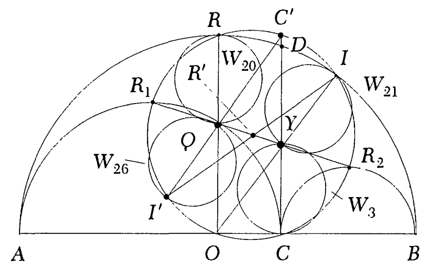

FIGURE 18
Circle 26.

circle (W_{20}) in line R_1R_2 and also symmetric to circle (W_{21}) in the center of circle $(OCIR)$. If CD cuts circle (R') again at C', then C', Q, and I' are collinear, as are also O, Y, and I. Finally, Schoch found one other circle (W_{27}), the smallest circle through point C and tangent to his circle (W_{15}), shown in FIGURE 19.

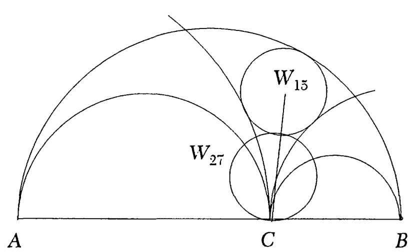

FIGURE 19
The Schoch circle 27.

In November of 1996 Paul Yiu **[7]** wrote a letter to Bankoff stating that he had just that morning discovered the circle I have called (W_{11}), adding "Maybe you have already known this. But isn't it wonderful?" He noted that its center is at the intersection of O_1F and O_2E **[8]**.

Woo's circles Peter Woo discovered an infinite family of Archimedean circles centered on line KV shown in FIGURES 11 and 12, which he called the *Schoch line*. In FIGURE 10 the Schoch circle (W_{15}) is the incircle of the curvilinear triangle CM_1M_2. The two circles $A(C)$ and $B(C)$, which pass through point C, whose centers lie on AB, and whose radii are twice the radii of circles (O_1) and (O_2) respectively, determine the arcs CM_1 and CM_2. Woo generalized this idea by using any positive multiple n instead of 2, leaving the circles to still pass through C, but moving their centers along line AB. Thus, as shown in FIGURE 20, draw two semicircles (O_n') and

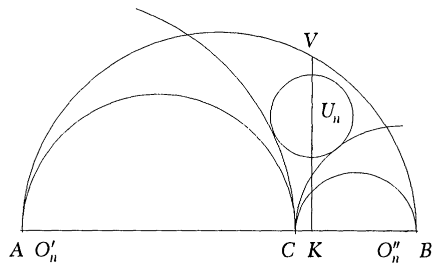

FIGURE 20
A typical Woo circle.

(O_n''), each tangent to line CD at point C, with centers O_n' on ray CA and O_n'' on ray CB, and with radii n times the radii of circles (O_1) and (O_2) respectively. Thus we call n the *radius multiplier*. Then the circle (U_n) with radius equal to that of the twin circles and tangent to (O_n') and (O_n'') will surprisingly have its center on the Schoch line. Conversely, any circle (U_n) with twin circle radius and centered sufficiently high up on the Schoch line will be tangent to two such circles (O_n') and (O_n'') for some positive real number n. The Woo circles are a generalization of Schoch's circles (W_{15}) and (W_{16}), also shown in FIGURES 10 and 11. FIGURE 24 shows selected Woo circles: (U_1) tangent to (O_1) and (O_2), $(U_2) = (W_{15})$ tangent to $A(C)$ and $B(C)$, (U_4), (U_7), and the limiting case $(U_0) = (W_{11})$.

Yiu's second circle When Paul Yiu read Woo's paper, he noted that circle $(W_{15}) = (U_2)$ was tangent internally to circle (O) and observed that there has to be a Woo circle (U_n) that is tangent to (O) externally. He proved that this circle, which we designate as (W_{28}), touches (O) at point D [8]. See FIGURE 27. He commented to me that "Archimedean circles start escaping the shoemaker's knife."

The proofs We now present proofs of some of our assertions. Let the radii and diameters of the circles (O), (O_1), and (O_2) be r and d, r_1 and d_1, and r_2 and d_2 respectively. Then, of course, $r = r_1 + r_2$ and $d = d_1 + d_2$. Let us denote the radius of each circle (W_i) by p_i. Although we shall not prove it, it is helpful in working with circles (W_{15}) and (W_{16}) and any of Woo's circles to know that

$$CK = \frac{r_1 r_2 (r_1 - r_2)}{(r_1 + r_2)^2}.$$

We shall need the fact that DD' of FIGURE 5 is equal to $2d_1d_2/(d_1 + d_2)$, the harmonic mean of the diameters of circles (O_1) and (O_2), so let us first display a delightful figure that shows this fact, along with some other means and their well-known relationship to one another. In the arbelos shown in FIGURE 21, OR is that radius of circle (O) that is perpendicular to the common diameter ACB. We use the notation above and that given in [3] for the means.

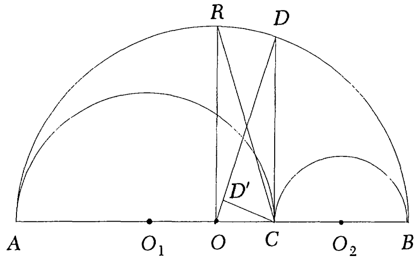

FIGURE 21
A mean figure.

THEOREM 1. *In* FIGURE 21 *we have that*

(i) *CR is the root-mean-square* $M_2(d_1, d_2)$ *of AC and CB.*
(ii) $OR = OD$ *is their arithmetic mean* $M_1(d_1, d_2)$.
(iii) *CD is their geometric mean* $M_0(d_1, d_2)$.
(iv) DD', *where* D' *is the foot of the perpendicular dropped from point C to OD, is their harmonic mean* $M_{-1}(d_1, d_2)$.
(v) *Finally,*

$$d_1 \geq M_2 \geq M_1 \geq M_0 \geq M_{-1} \geq d_2.$$

Furthermore, these inequalities are all strict when $d_1 \neq d_2$.

Proof. Of course, $AC = d_1$, $CB = d_2$, and $AB = d$, so $OD = OR = d/2 = (d_1 + d_2)/2 = M_1$. Now $OC = OB - CB = d/2 - d_2 = (d_1 - d_2)/2$, so by the Pythagorean theorem we have

$$CD = \sqrt{\left(\frac{d_1}{2} + \frac{d_2}{2}\right)^2 - \left(\frac{d_1}{2} - \frac{d_2}{2}\right)^2} = \sqrt{d_1 d_2} = M_0.$$

By similar right triangles, $DD'/CD = CD/OD$ and hence

$$DD' = \frac{CD^2}{OD} = \frac{d_1 d_2}{\frac{1}{2}(d_1 + d_2)} = \frac{2 d_1 d_2}{d_1 + d_2} = M_{-1}.$$

In the desired inequality we see that each internal member is the hypotenuse of a right triangle in which the next member is a leg. Thus triangle COR shows that $M_2 \geq M_1$, triangle OCD shows that $M_1 \geq M_0$, and triangle $CD'D$ produces $M_0 \geq M_{-1}$. Now

$$d_1 = AC = AO + OC = RO + OC \geq RC = M_2$$

and

$$M_{-1} = DD' = OD - OD' = OB - OD' \geq OB - OC = CB = d_2.$$

Equality occurs when and only when all three right triangles reduce to the same straight line segment, that is, when point C coincides with point O, when $d_1 = d_2$. □

Although Theorems 2 and 3 are readily and cleverly proved by inversion, as Bankoff showed in **[1]**, our proof will be by high school geometry.

THEOREM 2. *The radii p_1 and p_2 of circles* (W_1) *and* (W_2) *are equal to half the harmonic mean of r_1 and r_2. We denote this common value by p. That is,*

$$p_1 = p_2 = p = \frac{r_1 r_2}{r_1 + r_2} = \frac{r_1 r_2}{r}.$$

Proof. Draw O_1W_1 and OW_1 and drop perpendiculars from W_1 to line CD and to point Q on AB, as shown in FIGURE 22. Then

$$O_1W_1 = r_1 + p_1, \quad OW_1 = r - p_1 = r_1 + r_2 - p_1,$$
$$O_1Q = r_1 - p, \quad \text{and} \quad OQ = r_1 - r_2 - p_1.$$

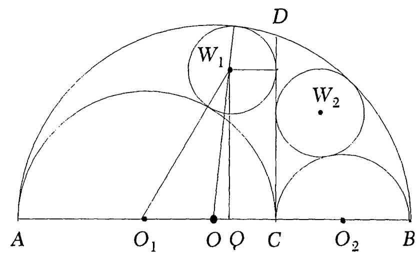

FIGURE 22
The twin circles.

From right triangles O_1W_1Q and OW_1Q we get that

$$QW_1^2 = (r_1 + p_1)^2 - (r_1 - p_1)^2 = (r_1 + r_2 - p_1)^2 - (r_1 - r_2 - p_1)^2,$$

which reduces to

$$4r_1 p_1 = 4(r_1 - p_1) r_2 \text{ and hence } p_1 = \frac{r_1 r_2}{r_1 + r_2} = p.$$

A similar argument shows that $p_2 = p$. □

THEOREM 3. *The radius p_3 of circle* (W_3) *is equal to p.*

Proof. Let r_3 denote the radius of circle (O_3), h the length of the perpendicular O_3H from O_3 to diameter ACB, and let $x = OH$. See FIGURE 23. For convenience we

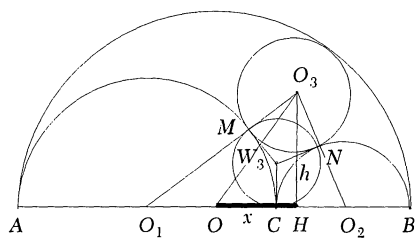

FIGURE 23
Triplet circle proof.

let $r = r_1 + r_2 = 1$, so that $p = r_1 r_2$. Then, from the three right triangles OO_3H, O_1O_3H, and O_2O_3H we obtain the three equations

$$x^2 + h^2 = (r_1 + r_2 - r_3)^2, \; (r_2 + x)^2 + h^2 = (r_1 + r_3)^2, \; \text{and} \; (r_1 - x)^2 + h^2 = (r_2 + r_3)^2.$$

Subtract the first equation from each of the other two, obtaining

$$2r_2^2 + 2r_2 x = -2r_1 r_2 + 4r_1 r_3 + 2r_2 r_3$$

and

$$2r_1^2 - 2r_1 x = -2r_1 r_2 + 2r_1 r_3 + 4r_2 r_3.$$

Now multiply the first of these two equations by r_1 and the second by r_2, and then add the resulting equations to get

$$4r_1{r_2}^2 + 4{r_1}^2 r_2 = 4{r_1}^2 r_3 + 4{r_2}^2 r_3 + 4r_1 r_2 r_3,$$

which we solve for r_3, finding that

$$r_3 = \frac{(r_1 + r_2) r_1 r_2}{{r_1}^2 + {r_2}^2 + r_1 r_2} = \frac{r_1 r_2}{1 - r_1 r_2}.$$

The sides of triangle $O_1O_2O_3$ have lengths $O_1O_2 = r_1 + r_2 = 1$, $O_3O_1 = r_3 + r_1$, and $O_2O_3 = r_2 + r_3$, so its semiperimeter is $1 + r_3$. By Heron's formula, its area K is given by

$$K^2 = (1 + r_3) r_1 r_2 r_3.$$

Since (W_3) is the incircle for that triangle, we also have

$$K = (\tfrac{1}{2})(r_1 + r_2) p_3 + (\tfrac{1}{2})(r_3 + r_1) p_3 + (\tfrac{1}{2})(r_2 + r_3) p_3 = (1 + r_3) p_3.$$

Equating the two expressions for K^2, we get that

$$(1 + r_3) r_1 r_2 r_3 = (1 + r_3)^2 {p_3}^2,$$

so that

$${p_3}^2 = \frac{r_1 r_2 r_3}{1 + r_3} = \frac{\dfrac{r_1^2 r_2^2}{1 - r_1 r_2}}{1 + \dfrac{r_1 r_2}{1 - r_1 r_2}} = r_1^2 r_2^2.$$

Hence $p_3 = r_1 r_2 = p$. It can be shown that $h = 2r_3$, an example of one of the delightful theorems presented in **[2]**. □

WOO'S THEOREM. *For any positive number n, draw two semicircles (O_n') and (O_n''), each tangent to line CD at point C, with centers O_n' on ray CA and O_n'' on ray CB, and with radii $r_1 n$ and $r_2 n$ respectively. Then the circle (U_n) with radius equal to that of the twin circles and externally tangent to (O_n'') and (O_n'') will have its center on the Schoch line. Conversely, any circle U_n with twin circle radius and centered on the Schoch line above height $2r_1 r_2 \sqrt{r_1 r_2}/(r_1 + r_2)^2$ will be tangent to two such circles (O_n') and (O_n'') for some nonnegative real number radius multiplier n. See* FIGURE 24.

Proof. Let C be the origin, ray CB the x-axis, and ray CD the y-axis. Choose the unit of length so that $r_1 + r_2 = 1$ and let the center of (U_n) have coordinates (x, y).

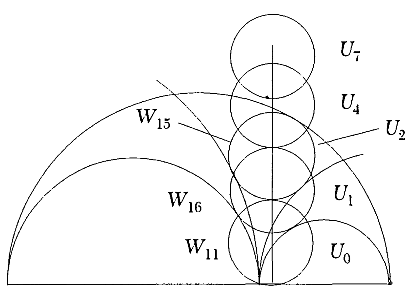

FIGURE 24
Selected Woo circles.

The radius of (U_n) is half the harmonic mean of r_1 and r_2, which is $r_1 r_2$ because $r_1 + r_2 = 1$. Then we have

$$O_n'U_n^2 - O_n''U_n^2 = (nr_1 + r_1 r_2)^2 - (nr_2 + r_1 r_2)^2 = (nr_1 + x)^2 - (nr_2 - x)^2,$$
$$2nr_1 r_2(r_1 - r_2) = 2n(r_1 + r_2)x,$$

and finally,

$$x = r_1 r_2 (r_1 - r_2),$$

which proves that U_n lies on the Schoch line. One can apply the Pythagorean theorem to triangle CKW_{11} to establish the minimum height requirement, and the converse is established. □

Some additional properties Now we cut short our proofs, having illustrated the techniques by which all circles can be shown to have the same radius. We conclude by stating a few more of the properties that these circles possess and locating one more circle.

Let T_i be the point of contact for circles (O_i) and (W_i) for $i = 1, 2$. Then $BD = BT_1$ and BT_1 is tangent to circles (O_1) and (W_1). Similarly, $AD = AT_2$ and AT_2 is tangent to circles (O_2) and (W_2). See FIGURE 25.

Earlier we stated that (W_6) and (W_7) were more than just translations of the twin circles onto the common diameter AB. We have seen that they are such translations also for (W_9) and (W_{10}), and for (W_{13}) and (W_{14}). Furthermore, as noted by Schoch and seen in FIGURE 13, they are semicircles of the inscribed similar arbelos. Also, FIGURE 26 shows that circle (W_6) is tangent to line AT_2, and circle (W_8) is tangent to

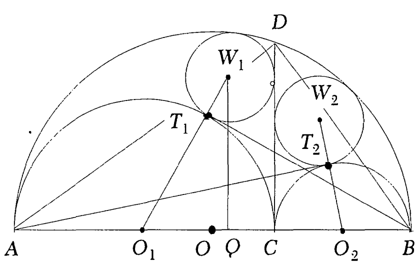

FIGURE 25
Twin circle tangents.

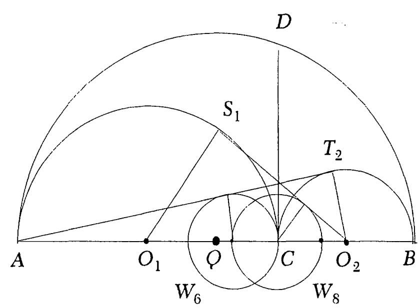

FIGURE 26
Tangents.

the line O_2S_1 drawn from O_2 tangent to circle (O_1). Similarly circle (W_7) is tangent to line BT_1, and circle (W_8) is tangent also to the line O_1S_2 drawn from O_1 tangent to circle (O_2). That is (W_6) is the circle through point C with center lying on segment AC and tangent to the line AT_2 and (W_7) is the circle through point C with center lying on segment BC and tangent to the line BT_1. Finally, (W_8) is the circle centered at point C and tangent to the two lines O_1S_2 and O_2S_1.

Our last circle is another Schoch circle. As shown in FIGURE 27, circles (W_5), (W_{12}), (W_4), and the second Yiu circle (W_{28}), that is, the Woo circle that is tangent externally to circle (O) at point D, all have centers that lie on line OD. Schoch discovered circle

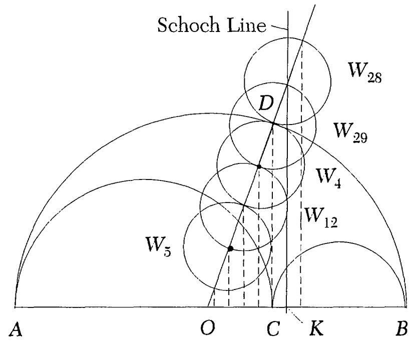

FIGURE 27
Yiu's circle 28 and Schoch's circle 29.

(W_{29}), the Archimedean circle centered at D, which has W_4W_{28} as diameter. Furthermore, the centers of all five of these circles project onto points on the base diameter AB that are spaced distance CK apart from one another, as shown by the dotted lines in the figure.

As a Woo circle (U_n), the second Yiu circle (W_{28}) has the value for its radius multiplier n given by

$$n = 2 + \frac{(r_1 + r_2)^2}{r_1 r_2}.$$

Conclusion Archimedes is credited with finding two delightful congruent circles in the arbelos, and Leon Bankoff opened a door by finding his triplet circle and later the quadruplet circle. Inspired by these masters, we have dramatically extended that family of circles. Although we have not stated as theorems and proved every property we indicated in our opening paragraphs, we have illustrated how to show that the circles (W_1) through (W_{29}) and the infinite family of Woo circles are Archimedean circles and do possess the claimed characteristics. We have achieved our goal of demonstrating that the twin circles of Archimedes are only two members of a huge family, in fact an infinite family, of congruent circles, all neatly hidden in that simple arbelos. When next you have new heels put on your shoes, you might describe some of these curious circles to your local cobbler.

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